hep-th/0010139

SOLVING N=3 SUPER-YANG-MILLS EQUATIONS IN HARMONIC SUPERSPACE[1]

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We analyze the superfield constraints of the D=4, N=3 SYM-theory using light-cone gauge conditions. The $SU(3)/U(1)\times U(1)$ harmonic variables are interpreted as auxiliary spectral parameters, and the transform to the harmonic-superspace representation is considered. Our nilpotent gauge for the basic harmonic superfield simplifies the SYM-equations of motion. A partial Grassmann decomposition of these equations yields the solvable linear system of iterative equations.

1 Introduction

It is well known that the geometric superspace constraints of the $D=4, N=3$ SYM-theory are equivalent to the equations of motion for the component fields [1]. The special projections of these superfield constraints can be interpreted as conditions of zero curvatures assosiated with the Grassmann covariant derivatives [2,3]. The possible connection of these zero-curvature conditions with integrability or solvability of the super-Yang-Mills (SYM) theories with 16 or 12 supercharges has been discussed more than 20 years in the framework of different superfield approaches (see, e.g. [2-6]). Here we shall use the harmonic-superspace approach [3] to analyze the solutions of the $N=3$ SYM-equations.

The harmonic-superspace (HSS) method has been introduced first for the solution of the $D=4,\ N=2$ off-shell superfield constraints [7]. Harmonic variables are analogous to twistor or auxiliary spectral variables used in integrable models. Harmonic and twistor methods give the explicit constructions of the general solutions to zero-curvature conditions in terms of independent functions on special *analytic* (super)spaces satisfying the generalized Cauchy-Riemann analyticity conditions. For instance, the off-shell $N=2$ superfields satisfy the conditions of the Grassmann (G-) analyticity.

In the standard harmonic formulation of the $D=4, N=2$ SYM-theory, the basic harmonic connection is G-analytic [7], and the 2-nd one (via the zero-curvature condition) appears to be a nonlinear function of the basic connection. The $N=2$ equation of motion is linearly dependent on the 2-nd harmonic connection, but it is the nonlinear equation for the basic connection [8]. It has been shown in Ref.[5] that one can alternatively choose the 2-nd harmonic connection as a basic superfield, so that the dynamical G-analyticity condition (or the Grassmann-harmonic zero-curvature condition) for the first connection becomes a new equation of motion. We shall use below a similar change of the basic HSS variables for the $N=3$ SYM-theory.

In the HSS-approach to the $D=4,\ N=3$ supersymmetry, the $SU(3)/U(1)\times U(1)$ harmonics have been used for the covariant reduction of the spinor coordinates and deriva-

[1] Talk at the 23 International Colloquium “Group-Theoretical Methods in Physics”, Dubna, 2000

tives and for the off-shell description of the SYM-theory in terms of the corresponding G-analytic superfields [3]. The $N = 3$ SYM-equations in the ordinary superspace have been transformed to the zero-curvature equations for the harmonic gauge connections, however, nobody tried earlier to obtain the general solution of these equations.

The alternative $SL(2, C)$-harmonic interpretation of the $N = 3$ SYM-equations and the corresponding harmonic zero-curvature equation for two G-analytic connections has been considered in Ref.[6]. We shall not discuss in this paper the problem of solving the equations of the $SL(2, C)$-harmonic approach.

An analogy with the self-dual SYM-solution [9] has been used recently for the analysis of the $10D$ SYM-solutions [10]. These authors have considered the very useful light-cone gauge conditions which break the space-time symmetry of the SYM-equations. We shall consider the analogous light-cone gauge conditions which preserve the $SU(3)$-invariance and allow us to use the harmonic approach.

It should be underlined that the $N = 3$ harmonic equations of motion contain three analytic connections, so they are more complicated than the $SU(2)$-harmonic equations using the single analytic connection. If one does not treat the $SU(3)$-harmonic connections as the basic variables, then the harmonic zero-curvature equations can be readily solved in terms of the non-analytic superfield matrix v (bridge). In this approach, the G-analyticity conditions for the composed harmonic connections can be considered as the dynamical equations for this independent superfield v.

It will be shown that the light-cone gauge conditions for the initial superfield connections correspond to the light-cone analyticity of the bridge matrix. We choose the nilpotent gauge for v and construct the nilpotent analytic harmonic connections in this representation. We demonstrate that the 1-st order harmonic bridge equations with the nilpotent connections produce also linear 2-nd order differential constraints for the basic matrices.

This talk is based on our work [11].

2 Reduced-symmetry representation of $D = 4,\ N = 3$ SYM constraints

The covariant coordinates of the $D = 4,\ N = 3$ superspace are

$$z^M = (x^{\alpha\dot{\alpha}}, \theta^\alpha_i, \bar{\theta}^{i\dot{\alpha}}) \ , \tag{1}$$

where $\alpha,\ \dot{\alpha}$ are the $SL(2, C)$ indices and $i = 1, 2, 3$ are indices of the fundamental representations of the group $SU(3)$.

We shall study solutions of the SYM-equations using the non-covariant representation and the new notation for these coordinates

$$\begin{aligned} &x^{\mp} \equiv x^{1\dot{1}} = t + x^3 \ , \quad x^{=} \equiv x^{2\dot{2}} = t - x^3 \ , \quad y \equiv x^{1\dot{2}} = x^1 + ix^2 \ , \quad \bar{y} \equiv x^{2\dot{1}} = x^1 - ix^2 \ , \\ &(\theta^+_i,\ \theta^-_i) \equiv \theta^\alpha_i \ , \quad (\bar{\theta}^{i+},\ \bar{\theta}^{i-}) \equiv \bar{\theta}^{i\dot{\alpha}} \ . \end{aligned} \tag{2}$$

based on the reduced symmetry $SO(1, 1) \times SU(3)$.

The $SO(1, 1)$ weights (helicities) of these coordinates are

$$w(x^{\mp}) = 2 \ , \quad w(x^{=}) = -2 \ , \quad w(y) = w(\bar{y}) = 0 \ , \quad w(\theta^\pm_i) = w(\bar{\theta}^{i\pm}) = \pm 1 \ . \tag{3}$$

The reduced representation of the algebra of $D=4,\ N=3$ spinor derivatives is

$$
\begin{array}{lll}
\{D^k_+, D^l_+\} = 0\ , & \{\bar{D}_{k+}, \bar{D}_{l+}\} = 0\ , & \{D^k_+, \bar{D}_{l+}\} = 2i\delta^k_l \partial_{\ddagger}\ , \\
\{D^k_-, D^l_-\} = 0\ , & \{\bar{D}_{k-}, \bar{D}_{l-}\} = 0\ , & \{D^k_-, \bar{D}_{l-}\} = 2i\delta^k_l \partial_{=}\ , \\
\{D^k_+, \bar{D}_{l-}\} = 2i\delta^k_l \partial_y\ , & \{D^k_-, \bar{D}_{l+}\} = 2i\delta^k_l \bar{\partial}_y\ , & \{D^k_+, D^l_-\} = \{\bar{D}_{k+}, \bar{D}_{l-}\} = 0\ .
\end{array} \tag{4}
$$

The last two relations can contain, in principle, six central charges, however, we shall consider the basic superspace without central charges. The general $N=3$ superspace has the odd dimension (6,6) in our notation with the reduced $SO(1,1)$ symmetry.

Let us consider the $(4|6,6)$-dimensional superspace gauge connections $A(z)$ and the corresponding covariant derivatives ∇

$$
\begin{array}{llll}
\nabla^i_\pm = D^i_\pm + A^i_\pm\ , & \bar{\nabla}_{i\pm} = \bar{D}_{i\pm} + \bar{A}_{i\pm}\ , & & \\
\nabla_{\ddagger} = \partial_{\ddagger} + A_{\ddagger}\ , & \nabla_{=} = \partial_{=} + A_{=}\ , & \nabla_y = \partial_y + A_y\ , & \bar{\nabla}_y = \bar{\partial}_y + \bar{A}_y\ .
\end{array} \tag{5}
$$

The $D=4,\ N=3$ SYM-constraints [1] have the following reduced-superspace form:

$$
\{\nabla^k_+, \nabla^l_+\} = 0\ , \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l+}\} = 0\ , \quad \{\nabla^k_+, \bar{\nabla}_{l+}\} = 2i\delta^k_l \nabla_{\ddagger}\ , \tag{6}
$$

$$
\{\nabla^k_+, \nabla^l_-\} = \bar{W}^{kl}\ , \quad \{\nabla^k_+, \bar{\nabla}_{l-}\} = 2i\delta^k_l \nabla_y\ , \tag{7}
$$

$$
\{\nabla^k_-, \bar{\nabla}_{l+}\} = 2i\delta^k_l \bar{\nabla}_y\ , \quad \{\bar{\nabla}_{k+}, \bar{\nabla}_{l-}\} = W_{kl}\ , \tag{8}
$$

$$
\{\nabla^k_-, \nabla^l_-\} = 0\ , \quad \{\bar{\nabla}_{k-}, \bar{\nabla}_{l-}\} = 0\ , \quad \{\nabla^k_-, \bar{\nabla}_{l-}\} = 2i\delta^k_l \nabla_{=}\ , \tag{9}
$$

where W_{kl} and $\bar{W}^{kl}$ are the gauge-covariant superfields constructed from the gauge connections. This reduced form of the 4D constraints is convenient for the study of dimensionally reduced SYM-subsystems in $D<4$.

Let us analyze first Eqs.(6) combined with the relations

$$
[\nabla^k_+, \nabla_{\ddagger}] = [\bar{\nabla}_{k+}, \nabla_{\ddagger}] = 0\ . \tag{10}
$$

The simplest light-cone gauge condition is

$$
A^k_+ = 0\ , \quad \bar{A}_{k+} = 0\ , \quad A_{\ddagger} = 0\ . \tag{11}
$$

The analogous gauge conditions excluding the part of connections has been considered in Ref.[9] for the self-dual $4D$ SYM-theory and in Ref.[10] for the 10D SYM equations.

Stress that the light-cone gauge preserves the $SU(3)$-symmetry of the $N=3$ superfield constraints, so we can use the harmonic-superspace method for non-trivial equations (7-9) in this gauge.

3 Harmonic-superspace equations for the nilpotent bridge matrix

The $SU(3)/U(1)\times U(1)$ harmonic superspace has been introduced in Ref.[3] for the off-shell description of the $4D$ $N=3$ SYM-theory. The dynamical equations in this approach have been transformed into the set of pure harmonic equations for the G-analytic superfield prepotentials. It should be stressed that nobody tried earlier to analyze in details the solutions of these harmonic equations.

Now we shall study the reduced-symmetry version of the $SU(3)/U(1)\times U(1)$ harmonic superspace which allows us to consider the non-covariant gauges and the dimensional reduction.

The $SU(3)/U(1)\times U(1)$ harmonics [3] parameterize the corresponding coset space. They form an $SU(3)$ matrix u^I_i and are defined modulo $U(1)\times U(1)$ transformations

$$u^1_i = u^{(1,0)}_i\ , \quad u^2_i = u^{(-1,1)}_i\ , \quad u^3_i = u^{(0,-1)}_i\ , \tag{12}$$

where the lower index i describes the triplet representation of $SU(3)$, and the upper indices 1, 2 and 3 correspond to different combinations of charges. The complex conjugated harmonics have opposite $U(1)$ charges and reverse positions of indices

$$u^i_1 = u^{i(-1,0)}\ , \quad u^i_2 = u^{i(1,-1)}\ , \quad u^i_3 = u^{i(0,1)}\ . \tag{13}$$

These harmonics satisfy the following relations:

$$\begin{aligned} &u^I_i u^i_J = \delta^I_J\ , \quad u^I_i u^k_I = \delta^k_i\ ,\\ &\varepsilon^{ikl}u^1_i u^2_k u^3_l = 1\ . \end{aligned} \tag{14}$$

The $SU(3)$-invariant harmonic derivatives act on the harmonics

$$\begin{aligned} &\partial^I_J u^K_i = \delta^K_J u^I_i\ , \quad \partial^I_J u^i_K = -\delta^I_K u^i_J\ ,\\ &[\partial^I_J, \partial^K_L] = \delta^K_J \partial^I_L - \delta^I_L \partial^K_J\ . \end{aligned} \tag{15}$$

We shall use the special $SU(3)$-covariant conjugation

$$\widetilde{u^1_i} = u^i_3\ , \quad \widetilde{u^3_i} = u^i_1\ , \quad \widetilde{u^2_i} = -u^i_2\ . \tag{16}$$

We can define the real analytic harmonic superspace $H(4,6|4,4)$ with 6 coset harmonic dimensions and the following left and right coordinates:

$$\begin{aligned} &\zeta = (X^{\mp\!\!\!\!+}\ ,\ X^{=}\ ,\ Y\ ,\ \bar{Y}\ ,\ \theta^\pm_2\ ,\ \theta^\pm_3\ ,\ \bar\theta^{1\pm}\ ,\ \bar\theta^{2\pm})\ ,\\ &X^{\mp\!\!\!\!+} = x^{\mp\!\!\!\!+} + i(\theta^+_3\bar\theta^{3+} - \theta^+_1\bar\theta^{1+})\ , \qquad X^{=} = x^{=} + i(\theta^-_3\bar\theta^{3-} - \theta^-_1\bar\theta^{1-})\ ,\\ &Y = y + i(\theta^+_3\bar\theta^{3-} - \theta^+_1\bar\theta^{1-})\ , \qquad \bar{Y} = \bar{y} + i(\theta^-_3\bar\theta^{3+} - \theta^-_1\bar\theta^{1+})\ , \end{aligned} \tag{17}$$

where $\theta^\pm_I = \theta^\pm_k u^k_I\ , \quad \bar\theta^{\pm I} = \bar\theta^{\pm k} u^I_k$.

The CR-structure in $H(4,6|4,4)$ involves the G-derivatives

$$D^1_\pm,\ \bar{D}_{3\pm}\ , \tag{18}$$

which commute with the harmonic derivatives D^1_2, D^2_3 and D^1_3.

These derivatives have the following explicit form in the analytic coordinates:

$$D^{(1,0)}_\pm \equiv D^1_\pm = \partial^1_\pm \equiv \partial/\partial\theta^\pm_1 \quad \bar{D}^{(0,1)}_\pm \equiv \bar{D}_{3\pm} = \partial_{3\pm} \equiv \partial/\partial\bar\theta^{3\pm}\ , \tag{19}$$

$$\begin{aligned} &D^{(2,-1)} \equiv D^1_2 = \partial^1_2 + i\theta^+_2\bar\theta^{1+}\partial_{\mp\!\!\!\!+} + i\theta^+_2\bar\theta^{1-}\partial_Y + i\theta^-_2\bar\theta^{1+}\bar\partial_Y + i\theta^-_2\bar\theta^{1-}\partial_{=}\\ &-\theta^+_2\partial^1_+ - \theta^-_2\partial^1_- + \bar\theta^{1+}\bar\partial_{2+} + \bar\theta^{1-}\bar\partial_{2-}\ ,\\ &D^{(-1,2)} \equiv D^2_3 = \partial^2_3 + i\theta^+_3\bar\theta^{2+}\partial_{\mp\!\!\!\!+} + i\theta^+_3\bar\theta^{2-}\partial_Y + i\theta^-_3\bar\theta^{2+}\bar\partial_Y + i\theta^-_3\bar\theta^{2-}\partial_{=}\\ &-\theta^+_3\partial^2_+ - \theta^-_3\partial^2_- + \bar\theta^{2+}\bar\partial_{3+} + \bar\theta^{2-}\bar\partial_{3-}\ ,\\ &D^{(1,1)} \equiv D^1_3 = \partial^1_3 + 2i\theta^+_3\bar\theta^{1+}\partial_{\mp\!\!\!\!+} + 2i\theta^+_3\bar\theta^{1-}\partial_Y + 2i\theta^-_3\bar\theta^{1+}\bar\partial_Y + 2i\theta^-_3\bar\theta^{1-}\partial_{=}\\ &-\theta^+_3\partial^1_+ - \theta^-_3\partial^1_- + \bar\theta^{1+}\bar\partial_{3+} + \bar\theta^{1-}\bar\partial_{3-}\ , \end{aligned} \tag{20}$$

where $\partial_{\ddagger} = \partial/\partial X^{\ddagger}$, $\partial_{=} = \partial/\partial X^{=}$, $\partial_{Y} = \partial/\partial Y$ and $\bar{\partial}_{Y} = \partial/\partial\bar{Y}$.

It is crucial that we start from the light-cone gauge conditions (11) for the $N = 3$ SYM-connections which break $SL(2,C)$ but preserve the $SU(3)$-invariance. Consider the harmonic transform of the covariant Grassmann derivatives in this gauge using the projections on the $SU(3)$-harmonics

$$\nabla^I_+ \equiv u^I_i D^i_+ = D^I_+ \,, \quad \bar{\nabla}_{I+} \equiv u^i_I \bar{D}_{i+} = \bar{D}_{I+} \,, \quad \{D^I_+, \bar{D}_{K+}\} = 2i\delta^I_K \partial_{\ddagger} \,, \tag{21}$$

$$\nabla^I_- \equiv u^I_i \nabla^i_- = D^I_- + \mathcal{A}^I_- \,, \quad \bar{\nabla}_{I-} \equiv u^i_I \nabla_{i-} = \bar{D}_{I-} + \bar{\mathcal{A}}_{I-} \,, \tag{22}$$

where the harmonized Grassmann connections $\mathcal{A}^I_-$ and $\bar{\mathcal{A}}_{I-}$ are defined.

The $SU(3)$-harmonic projections of the superfield constraints (7-9) can be derived from the basic set of the $N = 3$ G-integrability conditions for two components of the harmonized connection:

$$D^1_+ \mathcal{A}^1_- = \bar{D}_{3+} \mathcal{A}^1_- = D^1_+ \bar{\mathcal{A}}_{3-} = \bar{D}_{3+} \bar{\mathcal{A}}_{3-} = 0 \tag{23}$$

$$\begin{aligned} &D^1_- \mathcal{A}^1_- + (\mathcal{A}^1_-)^2 = 0 \,, \quad \bar{D}_{3-} \bar{\mathcal{A}}_{3-} + (\bar{\mathcal{A}}_{3-})^2 = 0 \,, \\ &D^1_- \bar{\mathcal{A}}_{3-} + \bar{D}_{3-} \mathcal{A}^1_- + \{\mathcal{A}^1_-, \bar{\mathcal{A}}_{3-}\} = 0 \,. \end{aligned} \tag{24}$$

All projections of the SYM-equations can be obtained by the action of harmonic derivatives D^I_K on these basic conditions.

These Grassmann zero-curvature equations have the very simple general solution

$$\mathcal{A}^1_-(v) = e^{-v} D^1_- e^v \,, \quad \bar{\mathcal{A}}_{3-}(v) = e^{-v} \bar{D}_{3-} e^v \,, \tag{25}$$

where *the bridge* v is the matrix in the Lie algebra of the gauge group. This superfield matrix satisfies the additional constraint

$$(D^1_+, \ \bar{D}_{3+}) v = 0 \,, \tag{26}$$

which is compatible with the light-cone representation (21). Thus, v does not depend on the Grassmann coordinates θ^+_1 and $\bar{\theta}^{3+}$.

Consider the gauge transformations of the bridge

$$e^v \ \Rightarrow \ e^\lambda e^v e^{\tau_r} \,, \tag{27}$$

where $\lambda \in H(4,6|4,4)$ is the (4,4)-analytic matrix parameter, and the parameter τ_r does not depend on harmonics. The matrix e^v realizes the harmonic transform of the gauge superfields $A^k_\pm, \bar{A}_{k\pm}$ in the central basis (CB) to the equivalent set of harmonic gauge superfields in the analytic basis (AB). We have partially fixed the CB-gauge invariance in (26), and the λ-gauge transformations of v will be used below.

The dynamical SYM-equations in the bridge representation (25) are reduced to the following harmonic differential conditions for the basic Grassmann connections:

$$(D^1_2, \ D^2_3, \ D^1_3) \left(\mathcal{A}^1_-(v), \ \bar{\mathcal{A}}_{3-}(v)\right) = 0 \,. \tag{28}$$

Using the off-shell $(4,4)$-analytic λ-transformations one can choose the non-supersymmetric nilpotent gauge condition for the superfield bridge

$$v = \theta^-_1 b^1 + \bar{\theta}^{3-} \bar{b}_3 + \theta^-_1 \bar{\theta}^{3-} d^1_3 \,, \quad v^2 = \theta^-_1 \bar{\theta}^{3-} [\bar{b}_3, b^1] \,, \quad v^3 = 0 \,, \tag{29}$$

$$e^{-v} = I - v + \frac{1}{2} v^2 = I - \theta^-_1 b^1 - \bar{\theta}^{3-} \bar{b}_3 + \theta^-_1 \bar{\theta}^{3-} (\frac{1}{2}[\bar{b}_3, b^1] - d^1_3) \,, \tag{30}$$

where the fermionic matrices $b^1, \bar{b}_3$ and the bosonic matrix d^1_3 are analytic functions of the coordinates ζ.

Note that the nilpotent gauges for the harmonic bridges and connections are possible for the harmonic formalisms with the off-shell analytic gauge groups only [3,7].

The conditions for the $SU(n)$-bridge are

$$\mathrm{Tr}\, v = 0 \,, \qquad v^\dagger = -v \,. \tag{31}$$

The matrices $b^1, \bar{b}_3$ and d^1_3 have the following properties in the gauge group $SU(n)$:

$$\mathrm{Tr}\, b^1 = 0 \,, \quad \mathrm{Tr}\, \bar{b}_3 = 0 \,, \quad \mathrm{Tr}\, d^1_3 = 0 \,, \tag{32}$$
$$(b^1)^\dagger = \bar{b}_3 \,, \quad (d^1_3)^\dagger = -d^1_3 \,. \tag{33}$$

Let us parameterize the Grassmann connection $\mathcal{A}^1_-(v)$ and $\bar{\mathcal{A}}_{3-}$ in terms of the basic analytic matrices $b^1, \bar{b}_3$ and d^1_3 (29)

$$\mathcal{A}^1_-(v) \equiv e^{-v} D^1_- e^v = b^1 - \theta^-_1 (b^1)^2 + \bar{\theta}^{3-} f^1_3 + \theta^-_1 \bar{\theta}^{3-} [b^1, f^1_3] \,, \tag{34}$$
$$\bar{\mathcal{A}}_{3-} \equiv e^{-v} \bar{D}_{3-} e^v = \bar{b}_3 + \theta^-_1 \bar{f}^1_3 - \bar{\theta}^{3-} (\bar{b}_3)^2 + \theta^-_1 \bar{\theta}^{3-} [\bar{f}^1_3, \bar{b}_3] \,, \tag{35}$$

where the following auxiliary superfields are introduced:

$$f^1_3 = d^1_3 - \frac{1}{2}\{b^1, \bar{b}_3\} \,, \quad \bar{f}^1_3 = -d^1_3 - \frac{1}{2}\{b^1, \bar{b}_3\} \,. \tag{36}$$

Equations $(D^1_2, D^2_3)\mathcal{A}^1_-(v) = 0$ generates the following independent relations for the (4,4)-analytic matrices:

$$D^1_2 b^1 = -\theta^-_2 (b^1)^2 \,, \tag{37}$$
$$D^2_3 b^1 = -\bar{\theta}^{2-} f^1_3 \,, \tag{38}$$
$$D^1_2 f^1_3 = \theta^-_2 [f^1_3, b^1] \,, \quad D^2_3 f^1_3 = 0 \,. \tag{39}$$

Equations $(D^1_2, D^2_3)\bar{\mathcal{A}}_{3-}(v) = 0$ are equivalent to the relations

$$D^1_2 \bar{b}_3 = \theta^-_2 \bar{f}^1_3 \,, \quad D^2_3 \bar{b}_3 = \bar{\theta}^{2-} (\bar{b}_3)^2 \,, \tag{40}$$
$$D^1_2 \bar{f}^1_3 = 0 \,, \quad D^2_3 \bar{f}^1_3 = \bar{\theta}^{2-} [\bar{b}_3, \bar{f}^1_3] \,. \tag{41}$$

In the case of gauge group $SU(n)$, the last equations are not independent, they can be obtained by conjugation from the equations for b^1 and f^1_3.

It is useful to derive the following relations for the matrices b^1 and $\bar{b}_3$ which do not contain the auxiliary matrices d^1_3, f^1_3 or $\bar{f}^1_3$:

$$\theta^-_2 D^1_2 (b^1, \ \bar{b}_3) = 0 \,, \quad \bar{\theta}^{2-} D^2_3 (b^1, \ \bar{b}_3) = 0 \,, \tag{42}$$
$$\theta^-_2 D^2_3 b^1 + \bar{\theta}^{2-} D^1_2 \bar{b}_3 = \theta^-_2 \bar{\theta}^{2-} \{b^1, \bar{b}_3\} \,. \tag{43}$$

Solutions of the linear equations for matrices f^1_3 and $\bar{f}^1_3$ satisfy the subsidiary condition

$$f^1_3 + \bar{f}^1_3 = -\{b^1, \bar{b}_3\} \,. \tag{44}$$

It is important that all these equations contain the nilpotent elements θ^-_2 or $\bar{\theta}^{2-}$ in the nonlinear parts, so they can be reduced to the set of linear iterative equations using the

partial Grassmann decomposition. In particular, the nilpotency of the basic equations yelds the subsidiary linear conditions for the coefficient functions

$$D^1_2 D^1_2(b^1, \bar{b}_3, d^1_3) = 0 \ , \quad D^2_3 D^2_3(b^1, \bar{b}_3, d^1_3) = 0 \ . \tag{45}$$

The harmonic linear equations for the analytic superfields $b^1, \bar{b}_3, d^1_3$ have simple (short) solutions with the finite number of the harmonic on-shell field components [2]. This harmonic shortness is an important restriction on the structure of the SYM-solutions.

4 Analytic representation of solutions

Remember that the following covariant Grassmann derivatives are flat in the analytic representation of the gauge group before the gauge fixing:

$$e^v \nabla^1_\pm e^{-v} \equiv \hat{\nabla}^1_\pm = D^1_\pm \ , \quad e^v \bar{\nabla}_{3\pm} e^{-v} \equiv \hat{\bar{\nabla}}_{3\pm} = \bar{D}_{3\pm} \tag{46}$$

The harmonic transform of the covariant derivatives using the matrix e^v (25) determines the composed on-shell harmonic AB-connections

$$\begin{aligned} &\nabla^I_K \equiv e^v D^I_K e^{-v} = D^I_K + V^I_K(v) \ , \\ &V^I_K(v) = e^v (D^I_K e^{-v}) \ . \end{aligned} \tag{47}$$

Note that the harmonic connections in the bridge representations satisfy automatically the harmonic zero-curvature equations.

It is evident that the basic equations (28) are equivalent to the following set of the dynamic G-analyticity relations for the composed harmonic connections:

$$(D^1_-, \ \bar{D}_{3-}) \left(V^1_2(v), V^2_3(v), V^1_3(v) \right) = 0 \ , \tag{48}$$

The positive-helicity analyticity conditions are satisfied automatically for the bridge in the gauge (26).

The dynamical G-analyticity equation (48) in gauge (29) is equivalent to the following harmonic differential bridge equation:

$$V^1_2(v) = \theta^-_2 b^1 \ , \qquad (V^1_2)^2 = 0 \tag{49}$$

$$D^1_2 e^{-v} = e^{-v} V^1_2 \ . \tag{50}$$

where the manifestly analytic nilpotent representation for the harmonic connection is used.

The 2-nd on-shell harmonic connection is also nilpotent $V^2_3(v) = -\bar{\theta}^{2-} \bar{b}_3$.

The spinor AB-connections can be calculated via the non-analytic harmonic connections by analogy with the $N = 2$ formalism of Ref.[8]

$$a^2_\pm = -D^1_\pm V^2_1 \ , \quad a^3_\pm = -D^1_\pm V^3_1 \ , \tag{51}$$

$$\bar{a}_{2\pm} = \bar{D}_{3\pm} V^3_2 \ , \quad \bar{a}_{1\pm} = \bar{D}_{3\pm} V^3_1 \tag{52}$$

where V^2_1, V^3_1 and V^3_2 are the non-analytic harmonic connections.

[2] The analogous short harmonic $N = 3, 4$ Abelian superfields have been considered in Ref.[13].

The 3-rd harmonic analytic connection can be readily calculated

$$V^1_3 = D^1_2 V^2_3 - D^2_3 V^1_2 + [V^1_2, V^2_3] = \theta^-_3 b^1 - \bar\theta^{1-}\bar b_3 \ , \tag{53}$$

where Eq.(43) is used.

The harmonic connection V^2_1 can be written in terms of the superfield b^1 only

$$V^2_1 = -\theta^-_1 D^2_1 b^1 \ . \tag{54}$$

The conjugated connection $V^3_2 = -(V^2_1)^\dagger$ contains matrix $\bar b_3$ only.

These connections satisfy the partial G-analyticity conditions

$$\bar D_{3\pm} V^2_1 = 0 \ , \quad D^1_\pm V^3_2 = 0 \ . \tag{55}$$

It is convenient to define the AB-superfield strengthes

$$\bar w^{12} = -D^1_+ D^1_- V^2_1 = -D^2_+ b^1 \ , \tag{56}$$

where Eq.(54) is used.

Stress that the single coefficient matrix b^1 generates the family of the AB-geometric objects: $V^1_2, V^2_1, a^2_\pm$ and $\bar w^{12}$. The conjugated $\bar b_3$-family of superfields contains $V^2_3, V^3_2, \bar a_{2\pm}$ and w_{23}.

The superfield $\bar w^{12}$ satisfy the (4,2)-dimensional G-analyticity conditions

$$D^1_\pm \bar w^{12} = \bar D_{3\pm}\bar w^{12} = D^2_\pm \bar w^{12} + [a^2_\pm, \bar w^{12}] = 0 \tag{57}$$

and the non-Abelian H-analyticity conditions

$$(\nabla^1_2, \nabla^2_3, \nabla^1_3)\bar w^{12} = 0 \ . \tag{58}$$

The (4,2)-analytic superspaces have been considered earlier in Refs.[12,13].

It should be noted that function d^1_3 (or f^1_3) is an auxiliary quantity in the framework of the analytic basis, since all harmonic and spinor connections and tensors of this basis can be expressed in terms of superfields b^1 and $\bar b_3$ only. Nevertheless, the construction of d^1_3 is important for the transition to the central basis.

Let us analyze the analytic equations for the basic fermionic (4,4) matrices b^1 and $\bar b_3$. The nonlinear terms in these equations contain the negative-helicity Grassmann coordinates $\theta^-_2, \theta^-_3, \bar\theta^{1-}$ and $\bar\theta^{2-}$, so the partial decomposition in terms of these coordinates is very useful for the iterative analysis of solutions. Equations for the auxiliary matrix d^1_3 do not give additional restrictions on b^1 and $\bar b_3$.

Consider first the decomposition of the harmonic derivatives and define the harmonic derivatives on the (4,0) analytic functions depending on the analytic Grassmann variables $\theta^+_2, \theta^+_3, \bar\theta^{1+}, \bar\theta^{2+}$

$$\begin{aligned}
\hat D^1_2 &= \partial^1_2 + i\theta^+_2\bar\theta^{1+}\partial_{\mp} - \theta^+_2\partial^1_+ + \bar\theta^{1+}\bar\partial_{2+} \ , \\
\hat D^2_3 &= \partial^2_3 + i\theta^+_3\bar\theta^{2+}\partial_{\mp} - \theta^+_3\partial^2_+ + \bar\theta^{2+}\bar\partial_{3+} \ , \\
\hat D^1_3 &= \partial^1_3 + 2i\theta^+_3\bar\theta^{1+}\partial_{\mp} - \theta^+_3\partial^1_+ + \bar\theta^{1+}\bar\partial_{3+} \ .
\end{aligned} \tag{59}$$

The (4,0) decomposition of the (4,4) analytic matrix functions has the following form:

$$b^1 = \beta^1 + \theta_2^- B^{12} + \theta_3^- B^{13} + \bar{\theta}^{1-} B^0 + \bar{\theta}^{2-} B_2^1 + \theta_2^- \theta_3^- \beta^0 + \theta_2^- \bar{\theta}^{1-} \beta^2 + \theta_3^- \bar{\theta}^{2-} \beta_2^{13}$$
$$+ \theta_3^- \bar{\theta}^{1-} \beta^3 + \bar{\theta}^{1-} \bar{\theta}^{2-} \beta_2 + \theta_2^- \bar{\theta}^{2-} \eta^1 + \theta_2^- \theta_3^- \bar{\theta}^{1-} B^{23} + \theta_3^- \bar{\theta}^{1-} \bar{\theta}^{2-} B_2^3 + \theta_2^- \theta_3^- \bar{\theta}^{2-} C^{13}$$
$$+ \theta_2^- \bar{\theta}^{1-} \bar{\theta}^{2-} C^0 + \theta_2^- \theta_3^- \bar{\theta}^{1-} \bar{\theta}^{2-} \eta^3 \ . \tag{60}$$

The analogous decompositions can be written for $\bar{b}_3$ and d_3^1.

It is easily to show that a part of the (4,0) coefficients can be constructed as the algebraic functions of the basic set of independent (4,0) matrices, for instance,

$$B_2^1 = -\hat{D}_2^1 B^0 - i\theta_2^+ \partial_Y \beta^1 \ , \tag{61}$$
$$B^{12} = \hat{D}_3^2 B^{13} + i\bar{\theta}^{2+} \bar{\partial}_Y \beta^1 \ , \tag{62}$$
$$\beta^2 = \hat{D}_3^2 \beta^3 - i\bar{\theta}^{2+} \bar{\partial}_Y B^0 \ , \tag{63}$$
$$\eta^1 = -\hat{D}_2^1 \beta^2 + i\bar{\theta}^{1+} \bar{\partial}_Y B^0 + i\theta_2^+ \partial_Y B^{12} + [\beta^1, B^0] \ . \tag{64}$$

The independent (4,0) matrix functions are

$$B^0, B^{13}, B^{23}, (C^0 - \bar{C}^0), B_2^3, \beta^1, \beta_2, \beta^3, \beta^0, \eta^3 \ . \tag{65}$$

The (4,0) matrix of the dimension $l = -1/2$ satisfies the linear equations

$$\hat{D}_2^1 \beta^1 = \hat{D}_3^2 \beta^1 = 0 \ . \tag{66}$$

The equations for the $l = -1$ matrices B^{13} and B^0 are

$$\hat{D}_2^1 B^{13} = 0 \ , \quad (\hat{D}_3^2)^2 B^{13} = 0 \ ,$$
$$\hat{D}_3^1 B^{13} = -2i\bar{\theta}^{1+} \bar{\partial}_Y \beta^1 - (\beta^1)^2 \ , \tag{67}$$
$$\hat{D}_3^2 B^0 = 0 \ , \quad (\hat{D}_2^1)^2 B^0 = 0 \ ,$$
$$\hat{D}_3^1 (B^0 - \bar{B}^0) = i\theta_3^+ \partial_Y \beta^1 + i\bar{\theta}^{1+} \bar{\partial}_Y \bar{\beta}_3 + \{\beta^1, \bar{\beta}_3\} \ . \tag{68}$$

The inhomogeneous linear equations for B^{13} and B^0 contain sources with the functions β^1 and $\bar{\beta}_3 = (\beta^1)^\dagger$ calculated on the previous stage.

The iterative equations for the (4,0) matrices with $l < -1$ can be analyzed analogously. Each independent iterative equation is manifestly resolved in terms of the harmonic derivatives of the corresponding function and the sources of these equations can be calculated on the previous stage of iteration.

Thus, it can be shown easily that the basic $N = 3$ harmonic equations for the (4,4) analytic moduli functions with the nilpotent nonlinear terms are equivalent to the finite number of the linear iterative (4,0) equations which contain non-Abelian sources constructed from the solutions of the previous step of iteration.

Note that all iterative equations are simplified essentially for the two-dimensional solutions which do not depend on variables Y and $\bar{Y}$.

5 Conclusions and acknowledgment

We have described the harmonic transform of the $N = 3$ SYM-equations of motion in the standard superspace to the differential HSS equations for the bridge matrix e^v which

connects different representations of the gauge superfields. The light-cone nilpotent gauge condition for matrix v simplifies significantly the analysis of the bridge equations. This condition yields the nilpotent harmonic-analytic gauge connections and the corresponding linear 2-nd order differential conditions for the basic matrices. The nilpotency of nonlinear terms in the basic HSS equations allow us to consider the simple iterative procedure based on the partial decomposition in Grassmann variables of the negative helicity. The finite set of these solvable linear iterative equations can be used for the explicit construction of the $N=3$ SYM-solutions in the harmonic superspace and the on-shell gauge superfields in the ordinary superspace.

Let us discuss shortly the problems of solving equations of the $N=0,1$ and 2 subsystems of the $N=3$ theory. One can study the problem of constructing solutions with the reduced number of fermion or scalar fields using the explicit constructions of the $N=3$ solutions, although the formal reduction of the stable and regular $N=3$ solution could, in principle, correspond to unstable or irregular solutions with lower supersymmetry.

The $N=2$ SYM-constraints together with equations of motion have the Grassmann-harmonic zero-curvature representation in the framework of the $SU(2)$-harmonic superspace [5]. It should be underline that the nilpotent bridge representation similar to (29) is also very useful for solving the $N=2$ SYM-equations. The corresponding $N=2$ solutions will be discussed elsewhere.

The author is grateful to J. Niederle for collaboration and to E. Ivanov and E. Sokatchev for discussions.

This work is supported by the Votruba-Blokhintsev programme in Joint Institute for Nuclear Research and also by the grants RFBR-99-02-18417, RFBR-DFG-99-02-04022 and NATO-PST.CLG-974874.

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